Problems and Solutions: 4th National Physics-Math Olympiad-2020

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2 Mathematics

Dhurmus Suntali Foundation

1. Dhurmus Suntali Foundation builds 100 houses in a row in a flood affected Mushahar settlement in Mahottari districts, which are to be painted red or orange. Dhurmus is quite particular and demands the painters that no three neighboring houses are all the same color.
(a) Find the maximum number of houses that can be painted orange. [4]
(b) In how many different ways may the houses be painted if exactly 67 are painted orange? [6]

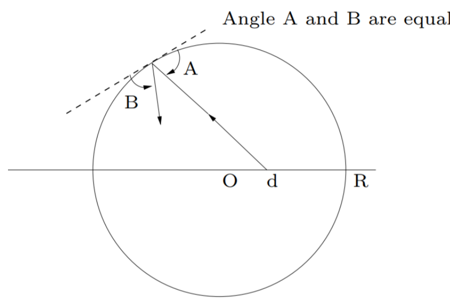
Solution
We shall denote a house that is to be painted red by R and a house that is to be painted orange by O.

(a) Let us number the houses 1 to 100 from left to right and consider the 34 blocks (1), (2, 3, 4), (5, 6, 7), . . ., (95, 96, 97), (98, 99, 100). As no three neighbouring houses can all be the same colour there must be a maximum of two orange houses in each of the 33 blocks of three houses. **From this we can deduce that at most 1 + 2 33 = 67 houses could be painted orange.** It is possible to paint exactly 67 houses orange, one colouring that achieves this is O followed by 33 blocks of ROO.

(b) Each block of three houses could be painted OOR,ORO or ROO. Note that the second colouring cannot be followed by the first and the third colouring cannot be followed by either the first or the second. This means that as soon as we choose the third colouring for one of our blocks of three houses then all successive blocks must have the same colouring. The first house must be painted O, as demonstrated in part (a), and the next block of three could be painted ORO or ROO. The only choice we have is when we first paint a block ROO, this could be in any of the 33 blocks of three houses or not at all. **This means there are 34 different ways to paint the 100 houses, which adheres to the strict demand of Dhurmus.**

Particle Inside the Circle

2. Consider a circle of radius R centered at the origin. A particle is "launched" from the x-axis at a distance, d, from the origin with 0 ¡ d ¡ R, and at an angle, , with the x-axis. The particle is reflected from the boundary of the circle so that the angle of incidence equals the angle of reflection. Determine the angle so that the path of the particle contacts the circle's interior at exactly eight points. Please note that should be determined in terms of the quantities R and d.



Solution

We note that since the angle of incidence stays the same with each reflection, the eight points must be spaced evenly around the circle. Denote these points (clockwise) by P1, P2, . . . P8. We note that to hit all 8 points, the path must cycle in of the following patterns since the number of points skipped between consecutive points visited is constant.

Patterns: a)P1P4P7P2P5P8P3P6, b)P1P2P3P4P5P6P7P8

These patterns cycle around and can also be taken in the opposite order.

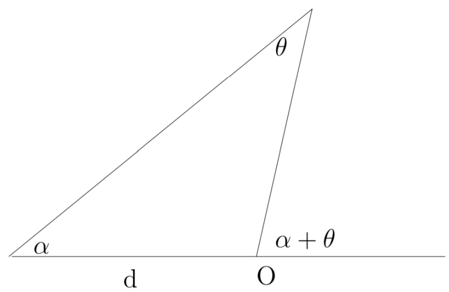
In pattern a, the angle of incidence is $\frac{1}{2} * \frac{2\pi}{8} = \frac{\pi}{8}$ radians

In pattern b, the angle of incidence is $\frac{1}{2} * \frac{6\pi}{8} = \frac{3\pi}{8}$ radians

We note that if the angle of incidence is θ and the particle is launched at angle α from point ($\alpha, 0)$ then the angle between the x-axis and the point at which the particle is reflected is $\theta + \alpha$, since the exterior angle equals the sum of the opposite interior angles. Now, we will find, given angle α and point $(\alpha, 0)$, what radius allows us to have angle of incidence θ. From above, we know that the intersection of the lines y = tan $\alpha(x + d)$ and $y = tan(\alpha + \theta)x$ lies on the circle.

Their intersection point can be shown to be:

$x_o = \frac{dtan\alpha}{tan(\alpha+\theta) - tan\alpha}$



$y_o = \frac{dtan\alpha tan(\alpha+\theta)}{tan(\alpha+\theta)-tan\alpha}$

Then $R^2 = x_o^2 + y_o^2$ since origin is (0,0). So,

$R = \frac{dtan\alpha}{tan(\alpha+\theta)-tan\alpha}\sqrt{1+tan^2(\alpha+\theta)}$

$R = \pm\frac{dtan\alpha sec(\alpha+\theta)}{tan(\alpha+\theta)-tan\alpha}$

$R = \pm\frac{dsin\alpha}{cos\alpha sin(\alpha+\theta)-sin\alpha cos(\alpha+\theta)}$

$R = \pm\frac{dsin\alpha}{sin\theta}$

Thus

$sin\alpha = \pm\frac{R}{d}sin\theta$. So

$\alpha = \pm\arcsin\left(\pm\frac{R}{d}sin\theta\right)$

Since θ can equal to $\frac{\pi}{8} or \frac{3\pi}{8}$

$\alpha = \arcsin(\pm\frac{R}{d}sin\frac{\pi}{8})$, or $\alpha = \arcsin(\pm\frac{R}{d}sin\frac{3\pi}{8})$

When both are defined, both solutions work, when only one is defined, only that solution works and when neither is defined, there are no solutions.

Function

3. A function f:$\mathbb{R} \to \mathbb{R}$ satisfies the following conditions:

i. $\mid f(a) - f(b) \mid \leq |a - b|$ for any real numbers $a, b \in \mathbb{R}$

ii. $f(f(f(0))) = 0$

Prove that f(0) = 0

[10]

Solution

We shall use the notation

$f^k(x) = f(f(...f(x)...))$

From,

$\mid f(0) \mid = \mid f(0) - 0 \mid \geq \mid f^2(0) - f(0) \mid \geq \mid f^3(0) - f^2(0) \mid = \mid f^2(0) \mid$

And

$\mid f^2(0) \mid = \mid f^2(0) - 0) \geq \mid f^3(0) - f(0) \mid = (0) \mid$

We have

$\mid f(0) \mid = \mid f^2(0) \mid$

There are two cases to consider. If f(0) = f^2(0),

$f(0) = f^2(0) = f^3(0) = 0$

Otherwise,

If $f(o) = -f^2(0)$

$\mid f(0) \mid = \mid f(0) - 0 \mid \geq \mid f^2(0) - f(0) \mid = 2 \mid f(0) \mid$

In both cases, $\mid f(0) \mid = 0$.

Therefore, f(0) = 0. Proved.