Problems and Solutions: 4th National Physics-Math Olympiad-2020

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1 Physics

A Naughty Brother

1. On a family picnic, you and your senior brother are taking alternate shots of football on a level ground.
a. Your brother kicks it with initial velocity u such that the area under the trajectory of the ball is maximum. Write the expression for time period, and determine the angle at which he kicked the ball. [3]
b. You happily bring back the ball as it is your turn now. At which maximum angle can you kick a ball so that its distance from you never decreases during the flight? [3]
c. After your kick, your brother orders you to bring back the ball. You reluctantly obey. As you reach the ball, you see your brother shouting to do it fast. Angered, you begin pumping out the air at the rate of 0.8 liters/s so that it is lose for your brother's kick. How soon will the pressure in the ball decrease by 100 times? (Assume no change in the ball's volume of 6 liters) [4]

Solution

a. Maximum Trajectory Area

let θ be the angle at which the ball was thrown. At any instant t, the coordinates of the ball's position are
x=ucosθt
y= usinθt-$\frac{1}{2}$ gt^2
time of flight (T) = $\frac{2usin\theta}{g}$
At small time dt, the horizontal displacement, $dx = ucos\theta dt$ Now,
The area under the trajectory can be obtained as below
A=$\int_0^T ydx$
A=$\int_0^T (usin\theta$t-$\frac{1}{2}$ gt$^2)ucos\theta dt$

Solving we get,

$A = \frac{2u^4 sin^3\theta}{3g^2}$

Differentiating with respect to θ and setting the derivative equal to zero, we get

$Tan\theta = \sqrt{3}$
$\therefore \theta = 60^\circ$ is the required angle at which your brother kicked the ball.

b. Ball's Distance Always Increasing

let θ be the required angle. At any instant t, the coordinates of the ball's position are $ucos\theta$t and usinθt-$\frac{1}{2}$ gt^2
If D is the distance from the point of projection, then
D$^2 = u^2cos^2\theta t^2 + u^2sin^2\theta$t$^2 - 2usin\theta t.\frac{1}{2}gt^2 + \frac{1}{4}g^2t^4$
$D^2 = u^2t^2 - usin\theta gt^3 + \frac{1}{4}g^2t^4$
Differentiating with respect to t and setting the derivative equal to zero we obtain,
$g^2t^2 - 3usin\theta gt + 2u^2 = 0$
It is the quadratic equation it t. If the determinant is zero or negative, the distance is always increasing. For the maximum angle, we set it equal to zero.
$(-3usin\theta g)^2 - 4.g^2.2u^2 = 0$
or, $sin\theta = \sqrt{\frac{8}{9}}$
$\therefore \theta = 70.53^\circ$ is the required angle.

c. Pressure Decreased by 100 Times

Given,
Volume of ball (V)= 6 liters
Pumping rate (C)= 0.8 liters/s
$\frac{P_i}{P_f} = \eta = 100$
let initially ρ be the density of gas. After small time dt, the density becomes $\rho + d\rho$, where $d\rho$ is essentially negative as the density decreases.
Using mass conservation,
$V.\rho = V.(\rho + d\rho) + Cdt.\rho$ [Since, V is constant]
or, Vd$\rho = -C\rho dt$
$or, \frac{d\rho}{\rho} = -\frac{C}{V}dt$

Since, $P \propto \rho$ [Assuming the process to be slow and isothermal]. Then,
$\frac{dP}{P} = -\frac{C}{V}dt$
Integrating,
$\int_{P_i}^{P_f} \frac{dP}{P} = -\frac{C}{V}\int_0^t dt$
$or, ln\frac{P_f}{P_i} = -\frac{C}{V}t$
$or, ln\frac{P_i}{P_f} = \frac{C}{V}t$
$or, ln\eta = \frac{C}{V}t$

$\therefore t = \frac{ln\eta . V}{C}$
Putting $\eta = 100$, V= 6 liters and C= 0.8 liters/s we get,

t = 34.54 seconds
$\therefore$ The pressure in the ball will decrease by 100 times in 34.54 seconds.

Proposal Went Wrong!

February 13:
You are in love with your junior who is 3 years younger than you. Tomorrow you are proposing her with a ring you have recently bought. Being a physicist, you are less focused on proposing skills and more on studying the ring.
a. You uniformly charged it with q, and wanted to place an electron at its center. So, you synthesized an electron with a beta minus (β^-) decay. Write the decay equation and calculate the energy released. [1]
b. Now your electron is at the center of the ring of radius a. After you displaced it by a small distance x along the axis (x$\ll$ a), find the expression for the frequency of oscillation. [3]
c. You have assumed the ring to be very thin. Dissatisfied, you measure its outer as well as the inner radius a and b respectively, and also weighed its mass. Then you noted the measurements as below and did some calculations:
M=0.015 $\pm$0.001 kg
a=18$\pm$1 mm
b=13$\pm$1 mm
What was the moment of inertia of the ring about an axis through the center and perpendicular to the plane of the ring? Also, present its estimated error. [1+2]

February 14:
Blimey, she rejected you! (You should have worked on your proposing skills) Broken, you wanted to leave this earth.
d. You travelled outward from earth for 2 years and back for another 2 years (both intervals as you measure them) with a constant speed parameter β of 0.997 (relative to earth). What will be the age difference between you and her after you return? [3]

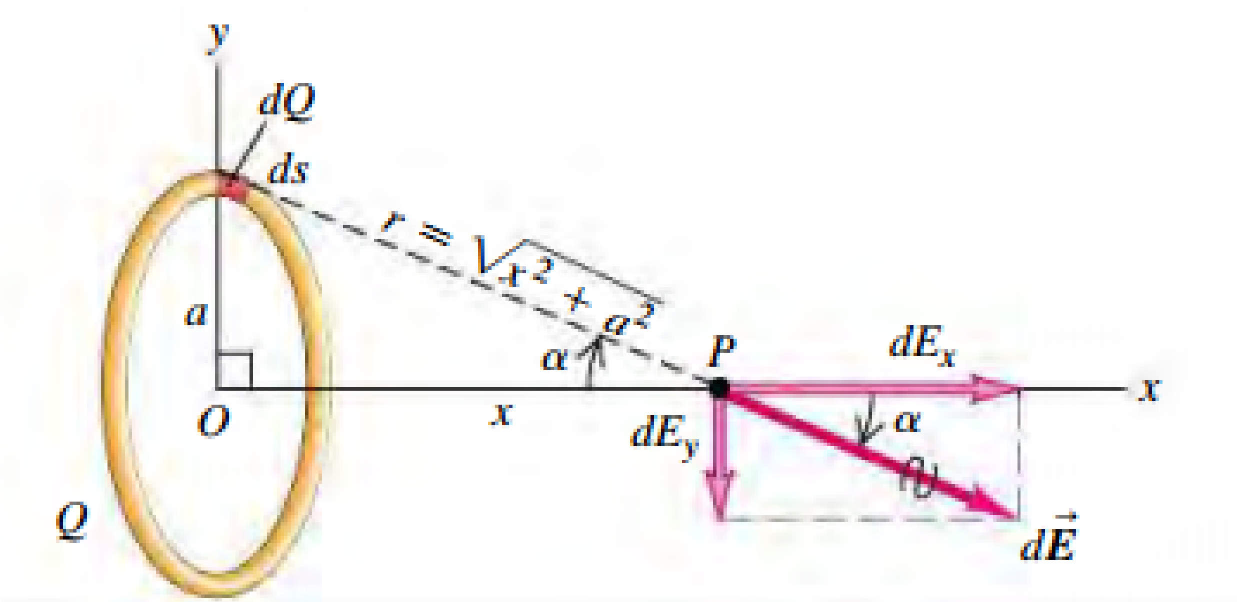
Solution

a. Decay Equation

In beta minus (β^-) decay, a neutron disintegrates into a proton, electron and a neutrino.
$n \rightarrow p + e^- + \overline{v}_e$
Energy Released = (Sum of masses in LHS - Sum of masses in RHS)* C^2

b. Frequency of Oscillation

Consider an element of the ring of length ds.
It contains a charge
$dQ = \frac{ds}{2\pi a} q$
So the electric field at a distance r is
$dE = \frac{ds}{2\pi a} q \frac{1}{4\pi\epsilon_o r^2}$



$dE = \frac{qds}{8\pi^2 a\epsilon_o r^2}$

The field dE acts in the direction α, and it can be resolved into a component dE cosα along the axis and dE sinα perpendicular to it. When we integrate round the entire ring, the perpendicular components will cancel out by symmetry. So, we only need to consider the components parallel to the axis. Thus,

E = $\frac{q}{8\pi^2 a\epsilon_o r^2} cos\theta \int_0^{2\pi a} ds$

$E = \frac{qcos\theta}{4\pi\epsilon_o r^2}$
Now we can express both $cos\theta$ and r^2 in terms of x and a

r$^2 = a^2 + x^2$

$cos\theta = \frac{x}{r} = \frac{x}{\sqrt{a^2+x^2}}$
So we can write E as a function of x as follows:

$E = \frac{qx}{4\pi\epsilon_o (a^2+x^2)^{3/2}}$
Considering the motion of an electron near the center of the ring, we can see that the electric

field E varies approximately linearly with x :

$E \approx \frac{qx}{4\pi\epsilon_o a^3}[Since, x \ll a]$
Since the electron has charge -e, the force acting upon it is given by

$F = -\frac{qe}{4\pi\epsilon_o a^3}x = m\frac{d^2x}{dt^2}$

This is the equation of simple harmonic motion, and hence angular frequency ω is given by

$\omega^2 = \frac{qe}{4\pi\epsilon_o a^3 m}$

Since $\omega = 2\pi f$, we get

$f = \sqrt{\frac{eq}{16\pi^2\epsilon_o a^3 m}}$, which is the required expression for the frequency of oscillation of electron.

c. Moment of Inertia and Error

Moment of Inertia of the ring (I) is

$$\frac{M}{2}(a^2 + b^2)$$

Using the value of M, a and b we obtain

$I = 3.69 * 10^{-6}$ kg m^2
The fractional error in I is given as

$\frac{dI}{I} = \sqrt{(\frac{\Delta M}{M})^2 + (\frac{(\Delta(a^2+b^2)}{a^2+b^2})^2}$

$\frac{dI}{I} = \sqrt{(\frac{\Delta M}{M})^2 + (\frac{\sqrt{(2a\Delta a)^2+(2b\Delta b)^2)}}{a^2+b^2})^2}$

d. Age Difference Between You and Her

$$A_{f(you)} - A_{i(you)} = 4 \tag{1}$$

$$A_{f(her)} - A_{i(her)} = 4\gamma \tag{2}$$

where,

$\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{1}{\sqrt{1-0.997^2}} = 12.91$

Let the age difference between you and her after you return to earth be x. Then
$A_{f(you)} = A_{f(her)} - x$ and
$A_{i(you)} = A_{i(her)} + 3$

Using these information in equation (1) we get,
$A_{f(her)} - x - A_{i(her)} - 3 = 4$
or, $4\gamma - x - 3 = 4$ [From equation 2]
or, x $= 4\gamma - 7 = 4 * 12.91 - 7$
$\therefore x = 44.68$

Thus, she will be 44.68 years older than you after your return. You definitely wouldn't want her now. See, physics heals everything!

Binary Star System

Two stars with masses M_1 and M_2 are in circular orbits around their center of mass. The star with mass M_1 has an orbit radius R_1 and the star with mass M_2 has an orbit radius R_2.
a. Prove that $R_1/R_2 = M_2/M_1$ [1]
b. Explain why the two stars have the same time period, and find the expression for time period in terms of R_1, R_2, M_1 and M_2. [1.5+2.5]
c. Find the distance between the components of the binary system in terms of total mass M and time period T. [3]
d. How much energy will be required to separate the two stars to infinity? [2]

Solution

a. Proof

let's assume the center of mass to be at origin. Then
$0 = \frac{M_1(-R_1)+M_2R_2}{M_1+M_2}$

$\therefore$R$_1/R_2 = M_2/M_1$ Proved.

b. Time Period

The force experienced by both the stars is same (since it is a mutual force) and this force is the centripetal force. So

$M_1\omega_1^2R_1 = M_2\omega_2^2R_2$
From above proof, M$_1R_1 = M_2R_2$
So
$\omega_1^2 = \omega_2^2$

$\therefore T_1 = T_2$

The force experienced by each stars is given by:

$F = \frac{GM_1M_2}{(R_1+R_2)^2}$

For star 1

$M_1\omega_1^2 R_1 = \frac{GM_1M_2}{(R_1+R_2)^2}$

or, $\frac{4\pi^2 R_1}{T^2} = \frac{GM_2}{(R_1+R_2)^2}$

$\therefore T^2 M_2 = \frac{4\pi^2 R_1(R_1+R_2)^2}{G}$

Similarly,

$T^2 M_1 = \frac{4\pi^2 R_2(R_1+R_2)^2}{G}$

Adding these two terms for time period, we get

$T^2(M_1 + M_2) = \frac{4\pi^2 (R_1+R_2)(R_1+R_2)^2}{G}$

$\therefore T = \frac{2\pi(R_1+R_2)^{3/2}}{\sqrt{G(M_1+M_2)}}$, which is the required expression for the time period of binary star system.

c. Distance Between the Components

From above expression of time period

$T = \frac{2\pi(R_1+R_2)^{3/2}}{\sqrt{G(M_1+M_2)}}$

So

$(\mathrm{R}_1 + R_2)^{3/2} = \frac{T\sqrt{G(M_1+M_2)}}{2\pi}$

$\therefore R_1 + R_2 = \sqrt[3]{\frac{GMT^2}{4\pi^2}}$ is the required distance between the components of the binary star system, where $\mathrm{M}=\mathrm{M}_1 + M_2$.

d. Energy Required to Separate the Stars to Infinity

Let E be the energy required to separate the stars to infinity. Then,

$K_1 + U_1 + E = K_2 + U_2$

At infinity, $K_2 = 0$ and $U_2 = 0$

So

$\mathrm{E} = \text{-}\mathrm{K}_1 - U_1$

$\therefore E = -\frac{1}{2}M_1(\frac{2\pi R_1}{T})^2 - \frac{1}{2}M_2(\frac{2\pi R_2}{T})^2 + \frac{GM_1M_2}{(R_1+R_2)}$ is the required expression for energy required to separate the binary stars to infinity.